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Dilute Antiferromagnets: Imry–Ma Argument, Hierarchical Model, and Equivalence to Random Field Ising Models

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We discuss some aspects of the problem of the equivalence of dilute antiferromagnets and random field Ising models. We first investigate for dilute antiferromagnets the validity of the arguments of Imry and Ma. It turns out that they are applicable, but some care is required concerning the role played by the so-called internal Peierls contours. Next we consider a hierarchical version of a dilute antiferromagnetic Ising model in the presence of a uniform magnetic field and show that a renormalization group transformation maps it exactly into a hierarchical version of the random field Ising model, thus proving their equivalence as far as the critical behavior is concerned. In particular this implies that phase transition with spontaneous magnetization occurs only for dimension $d > 2$. Finally we show that in the absence of internal Peierls contours both models, in their hierarchical versions, exhibit phase transition already in dimension $d = 2$.

KEY WORDS: Random fields; dilute antiferromagnets; Imry–Ma argument.

1. INTRODUCTION

Dilute antiferromagnets in the presence of a uniform external magnetic field have been conjectured to exhibit the same critical behavior as Ising models in the presence of external random magnetic fields.[1] There are, however, only a few rigorous results in the direction of establishing the claimed

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equivalence. In the context of Curie–Weiss models this equivalence has been established by Amaro de Matos *et al.*[2] both at the level of thermodynamics and of fluctuations which are nontrivial at the criticality.

In the random field Ising model, the heuristics for the computation of the lower critical dimension is provided by a beautiful argument of Imry and Ma[3] concerning the typical energy balance associated with a Peierls contour. This argument by itself is not sufficient for a proof of a phase transition in dimension $d \geqslant 3$, since it does not take into account the existence of the so-called internals Peierls contours (i.e., contours inside contours). A complete proof was finally provided by Bricmont and Kupiainen[4] with renormalization group techniques being the extra ingredient. The key ideas of this technical work were highlighted in ref. 5, where a hierarchical version of the model was discussed. This model incorporated the physics of the Imry–Ma argument and at the same time simplified the renormalization group steps necessary to control the effect of the internal contours.

In this paper we first consider the problem of the validity of Imry–Ma argument as applied to dilute antiferromagnets. The conclusion is that they are applicable, but some care is to be exercised in the bond dilute model: before applying the argument, one should first remove those sites which, in consequence of the dilution, are completely decoupled from the rest of the system. Without removal of these sites, a simplified model which excludes internal contours shows phase transition with spontaneous magnetization already in dimension $d = 2$, with an energy balance which does not exhibit the Imry–Ma competition.

Next we introduce a hierarchical version of the bond dilute model with the same structure of the hierarchical version of ref. 5. For this model we show that a one-step renormalization group transformation maps it exactly into the Bricmont–Kupiainen model, thus establishing their equivalence. In particular, the two models have lower critical dimension equal to 2, identical asymptotic behavior of correlation functions, and identical critical exponents. Finally we revisit the hierarchical random field Ising model with the extra exclusion of internal contours and show that in this situation the model has a phase transition already in dimension $d = 2$. This indicates that the two-dimensional model (for which Aizenman and Wehr[6] proved absence of spontaneous magnetization and where the Imry–Ma argument is inconclusive) is sensitive to the presence of internal contours, even in the hierarchical approximation, for which Bricmont and Kupiainen[5] showed absence of spontaneous magnetization when they are included.

Parts of this work appeared in ref. 7.

2. THE IMRY–MA ARGUMENT IN DILUTE ANTIFERROMAGNETS

A bond dilute antiferromagnetic Ising model in the presence of an external uniform magnetic field h in an finite volume $\Lambda \subset \mathbf{Z}^d$ with free boundary conditions is given by the energy function of a configuration $\sigma = \{\sigma_i, i \in \mathbf{Z}^d\}$:

$$H_\Lambda^{\mathrm{DAF}}(\sigma) = J \sum_{\langle ij \rangle \in \Lambda} \xi_{ij} \sigma_i \sigma_j + h \sum_{i \in \Lambda} \sigma_i \tag{2.1}$$

where $J > 0$, and the dilution is described by the independent identically distributed random variables

$$\xi_{ij} = \begin{cases} 1 & \text{with probability } 1 - p \\ 0 & \text{with probability } p \end{cases} \tag{2.2}$$

In order to introduce Peierls contours for this model we first perform the transformation

$$\sigma_i \to (-1)^{\|i\|} \sigma_i \tag{2.3}$$

where $\|i\| = \sum_{k=1}^d i_k$, with the notation $i = (i_1, \ldots, i_d)$. The transformation (2.3) which inverts the spin variables in the odd sublattice of $\mathbf{Z}^d$ maps the model (2.1) into a dilute ferromagnetic model in the presence of a staggered magnetic field:

$$H_\Lambda(\sigma) = -J \sum_{\langle ij \rangle \in \Lambda} \xi_{ij} \sigma_i \sigma_j + h \sum_{i \in \Lambda} (-1)^{\|i\|} \sigma_i \tag{2.4}$$

Let us now investigate how Imry–Ma arguments could be applied to the model (2.4). So, consider the model with boundary condition $+$, i.e., $\sigma_i = +1$, for all i in the external boundary of Λ. For a given configuration σ we draw Peierls contours by the usual prescription so that a one-to-one correspondence between σ and a collection Γ of closed nonintersecting contours γ is established. The energy associated with Γ can be written apart from an additive constant as

$$H_\Lambda(\Gamma) = 2J \sum_{\gamma \in \Gamma} \sum_{\langle ij \rangle \in \gamma} \xi_{ij} + h \sum_{i \in \Lambda} (-1)^{\|i\|} \sigma_i(\Gamma) \tag{2.5}$$

where $\sigma_i(\Gamma) = +1$ if the number of contours in Γ containing the site i is even, and $\sigma_i(\Gamma) = -1$ otherwise. If a given configuration σ is such that

there are no contours in Γ which are contained inside other contours, then the contribution of the second term in (2.5) may be explicitly computed:

$$H_A(\Gamma) = \sum_{\gamma \in \Gamma} \left(2J \sum_{\langle ij \rangle \in \gamma} \xi_{ij} - h \sum_{i \in \gamma} (-1)^{\|i\|} \right) \tag{2.6}$$

If we now take (2.6) as the energy function of a model whose dynamical variables are collections of contours $\Gamma = \{\gamma_i\}$ where the γ_i are closed contours which are not inside each other, i.e., no internal contours, it is possible to provide a Peierls argument to prove spontaneous magnetization for $d \geqslant 2$. In fact, the typical contribution to the energy coming from a given contour γ is bigger than or equal to

$$[2J(1-p) - h]\,|\gamma| \tag{2.7}$$

where by $|\gamma|$ we mean the length (or area in $d=3$) of the contour γ. To make this heuristic argument rigorous we use the usual Peierls argument technique to get the immediate bound for the probability that σ_0 is negative:

$$\overline{\mathbf{P}\{\sigma_0 = -1\}} \leqslant \sum_{s=0}^{\infty} \#(s) (\overline{\exp\{-2J\beta\xi\}})^s \exp\{hs\} \tag{2.8}$$

where $\#(s)$ denotes the number of contours containing the origin with "surface" s, and where we used the independence of the random variables ξ_{ij}. The bar means average with respect to the dilution. With the usual bound $\#(s) \leqslant s^{d/(d-1)}(2d)^s$ and from $\overline{\exp\{-2J\beta\xi\}} = p + (1-p)\exp(-2J\beta)$ it follows that we have spontaneous magnetization provided p, h are sufficiently small and β is sufficiently large if the dimension $d \geqslant 2$. This shows that in the absence of internal contours there are no strong fluctuations of the type described in the arguments of Imry and Ma.

The above result is misleading, however, for the properties of the full model as we now discuss. The imposition of absence of internal contours should be imposed only *after* removing from the system sites which are statistically independent. This is done as follows. We first remark that if in a given realization of the dilution $\xi = \{\xi_{ij}\}$ a given site $i \in \mathbf{Z}^d$ is such that $\xi_{ij} = 0$ for all nearest neighbor sites j, then the site i is completely decoupled from the rest of the system and it contributes trivially to the free energy and correlation functions. It is then convenient to eliminate the dynamical variable σ_i and view the full model as a spin system in $\mathbf{Z}^d \backslash \{i\}$, i.e., the system has a hole at the site i. The probability of a hole at any site is $q = p^{2d}$, but two nearest neighbor sites are not statistically independent as far as the probability of being holes is concerned. But if the dilution p is

sufficiently small, holes will typically be far apart from each other and it will be a good approximation to consider a new model with energy function given by

$$H'_\Lambda(\sigma) = -J \sum_{\langle ij \rangle \in \Lambda} \xi_{ij}\sigma_i\sigma_j + h \sum_{i \in \Lambda} (-1)^{\|i\|} \eta_i \sigma_i \tag{2.9}$$

where $\eta = \{\eta_i\}$ are independent identically distributed random variables with

$$\eta_i = \begin{cases} 1 & \text{with probability } q \\ 0 & \text{with probability } (1-q) \end{cases} \tag{2.10}$$

This new model is clearly similar to the site dilute model:

$$H^s_\Lambda(\sigma) = -J \sum_{\langle i,j \rangle \in \Lambda} \xi_i\xi_j\sigma_i\sigma_j + h \sum_{i \in \Lambda} (-1)^{\|i\|} \xi_i \sigma_i \tag{2.11}$$

where

$$\xi_i = \begin{cases} 1 & \text{with probability } 1-p \\ 0 & \text{with probability } p \end{cases} \tag{2.12}$$

Let us consider, for the site dilute model, a contour γ with no internal contours. In this model the contribution to energy from γ is

$$\varepsilon(\gamma) = 2J \sum_{\langle ij \rangle \in \gamma} \xi_i\xi_j - h \sum_{i \in \gamma} (-1)^{\|i\|} \xi_i \tag{2.13}$$

The mean value of the random variable $\varepsilon(\gamma)$ is

$$\overline{\varepsilon(\gamma)} = 2J(1-p)^2|\gamma| - h(1-p) \sum_{i \in \gamma} (-1)^{\|i\|} \tag{2.14}$$

and so $\overline{\varepsilon(\gamma)}$ is of the order of $|\gamma|$. Its variance is approximately given by

$$\{\overline{[\varepsilon(\gamma) - \overline{\varepsilon(\gamma)}]^2}\}^{1/2} = 2J|\gamma|^{1/2}(1-p)^2 p(2-p) + h[v(\gamma)]^{1/2} p(1-p) \tag{2.15}$$

where $v(\gamma)$ is the volume (area in $d=2$) of the region enclosed by γ. So for large contours the variance is dominated by the term proportional to $[v(\gamma)]^{1/2} \simeq h|\gamma|^{d/2(d-1)}$, which will compete with a term proportional to $2J|\gamma|$ coming from the average. This is the Imry–Ma energy balance for the model with site dilution (2.11). Therefore the heuristics for the phase transition mechanism in this model is the same as for the random field Ising model and it should be possible to provide a rigorous proof of this fact with the same methods used in ref. 4. From the above discussion it follows that the same arguments apply also to the model with bond dilution.

3. THE HIERARCHICAL MODEL

Hierarchical models are tailor-made for the application of renormalization group techniques.[8,9] The version of the hierarchical model for the bond dilute system we shall use is the one introduced in ref. 5 for the random field Ising model.

If L is an even positive integer we consider for each $n=0, 1, 2,...$ the sublattices

$$(L^n\mathbf{Z})^d = \{L^n x, x \in \mathbf{Z}^d\}$$

with lattice spacing L^n. A configuration Γ of the system is given by

$$\Gamma = \{\gamma_n(x), x \in (L^n\mathbf{Z})^d, n=0, 1, 2,...\} \tag{3.1}$$

where $\gamma_n(x) \in \{0, 1\}$. When $\gamma_n(x)=1$, it indicates the presence of a contour with the shape of a square with side L^n along the coordinate axis, centered at the site $x+[(L^n-1)/2]e$, where $x \in (L^n\mathbf{Z})^d$ and $e=(1, 1,..., 1) \in \mathbf{Z}^d$. The alternative choice of an odd L, although leading to the same qualitative results, is less convenient, as will be shown later.

If we enclose the system in a finite box $\Lambda \subset \mathbf{Z}^d$ of side L^N with center at the point $[(L^N-1)/2]e$, the energy of a configuration Γ is given by

$$H_N(\Gamma) = 2J \sum_{n=0}^{N} L^{n(d-1)} \sum_{x \in \Lambda_n} \xi_n(x)\gamma_n(x) + \sum_{x \in \Lambda} h_x \sigma_x(\Gamma) \tag{3.2}$$

where $\Lambda_n = \Lambda \cap (L^n\mathbf{Z})^d$, $\Lambda_0 \equiv \Lambda$, and

$$\sigma_x(\Gamma) = (-1)^{p_x(\Gamma)} \tag{3.3}$$

where $p_x(\Gamma)$ is the number of contours in Γ enclosing the site $x \in \Lambda$. The external field $h_x = (-1)^{\|x\|}h$ is staggered since we are working with the ferromagnetic model. This definition for $\sigma_x(\Gamma)$ amounts to choosing the spins at the boundary of Λ to be $+1$. Finally, the random variables $\{\xi_n(x), x \in (L^n\mathbf{Z})^d, n=0, 1, 2,...\}$ describe the dilution; they are independent and

$$\xi_n(x) = \begin{cases} 1 & \text{with probability } 1-p_n \\ 0 & \text{with probability } p_n \end{cases}$$

For simplicity we shall restrict our discussion to the case where $p_n=0$, for $n \geqslant 1$, i.e., only the contours of the first hierarchy $n=0$ are diluted. Using the decomposition

$$\Gamma = \bigcup_{n \geqslant 0} \Gamma_n = \Gamma_0 \cup \left(\bigcup_{n \geqslant 1} \Gamma_n \right) \tag{3.4}$$

where $\Gamma_n = \{\gamma_n(x),\ x \in (L^n\mathbf{Z})^d\}$, we get from (3.2)

$$H_N(\Gamma) = 2J \sum_{n=1}^{N-1} L^{n(d-1)} \sum_{x \in \Lambda_n} \gamma_n(x) + V(\Gamma) \tag{3.5}$$

where

$$V(\Gamma) = \sum_{x \in \Lambda} \{2JL^{(d-1)}\, \xi_0(x)\, \gamma_0(x) + h_x \sigma_x(\Gamma)\} = \sum_{x \in \Lambda_1} v_x \tag{3.6}$$

with

$$v_x = \sum_{y \in B_x^{(1)}} \{2JL^{(d-1)} \xi_0(y)\, \gamma_0(y) + h_y \sigma_y(\Gamma)\} \tag{3.7}$$

Here

$$B_x^{(1)} = \left\{ Lx + \sum_{i=1}^{d} n_i e_i,\ n_i = 0,\ldots,\ (L-1) \text{ for } i = 1,\ldots,\ d \right\} \tag{3.8}$$

where e_i, $i = 1,\ldots,\ d$, are the unit lattice vectors. That is, $B_x^{(1)}$ is the block indexed by x. We are now in a position to apply the renormalization group transformation. We compute the partition function

$$Z_N = \sum_{\Gamma} \exp\{-\beta H_N(\Gamma)\} \tag{3.9}$$

in two steps, by first summing over Γ_0 with $\Gamma' = (\Gamma_1,\ldots,\ \Gamma_N)$ fixed. We define $V'(\Gamma')$ through

$$\exp[-V'(\Gamma')] \equiv \sum_{\Gamma_0} \exp[-\beta V(\Gamma)] \tag{3.10}$$

The hierarchical structure of the models allows it to be explicitly computed:

$$V'(\Gamma') = \sum_{x \in \Lambda_1} \{h'_x \sigma_x(\Gamma') + C_x\} \tag{3.11}$$

where the function h'_x is given by

$$e^{2h'_x} = \prod_{y \in B_x^{(1)}} \left(\frac{e^{\beta h_y} + e^{-\beta \xi_y} e^{-\beta h_y}}{e^{-\beta h_y} + e^{-\beta \xi_y} e^{+\beta h_y}} \right) \tag{3.12}$$

and C_x is an irrelevant random (since it depends on ξ_y, $y \in B_x^{(1)}$) constant, as it does not depend on Γ'. Here we used the identity

$$\sigma_x(\Gamma) = \sigma_x(\Gamma')(-1)^{\gamma_x} \tag{3.13}$$

Further manipulation of (3.12) leads to

$$2h'_x = \sum_{y \in B_x^{(1)}} \xi_y \ln\left(\frac{e^{\beta h_y} + e^{-\beta} e^{-\beta h_y}}{e^{-\beta h_y} + e^{-\beta} e^{+\beta h_y}}\right) = g_\beta(h) \sum_{y \in B_x^{(1)}} (-1)^y \xi_y \quad (3.14)$$

where

$$g_\beta(h) = \ln\left(\frac{e^{\beta h} + e^{-\beta} e^{-\beta h}}{e^{-\beta h} + e^{-\beta} e^{+\beta h}}\right) \quad (3.15)$$

Formula (3.14) expresses each h'_x as a sum of independent random variables and shows that they are independent identically distributed with mean and variance given by

$$\begin{aligned} \overline{h'_x} &= 0 \\ \overline{h'^2_x} &= g_\beta(h)^2 L^d \end{aligned} \quad (3.16)$$

Therefore we have as a net effect of the renormalization group transformation the identity

$$Z_N = C Z_{N-1}^{\mathrm{RF}} \quad (3.17)$$

where $C = \exp(\sum_{x \in \Lambda_1} C_x)$ and Z_N^{RF} denotes the partition function of a hierarchical random field Ising model of the type introduced by Bricmont and Kupiainen[5] with energy function given by

$$H_N^{\mathrm{RF}}(\Gamma) = 2J \sum_{n=0}^{N} L^{n(d-1)} \sum_{x \in \Lambda_n} \gamma_n(x) + \sum_{x \in \Lambda} h'_x \sigma_x(\Gamma) \quad (3.18)$$

with the random field with h'_x given by (3.12).

We are now in a position to derive the relation between correlation functions in the two models. Let us consider the dilute antiferromagnet, an observable $F(\Gamma') = F(\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$, $k \leqslant N$, which does not depend on Γ_0, the configuration of contours of the first hierarchy. For this observable it follows from (3.17) that, for a given realization ξ of the dilution variables, its Gibbs expectation value is given by

$$\langle F \rangle_N^{\mathrm{DAF}} = \langle \tilde{F} \rangle_{N-1}^{\mathrm{RF}} \quad (3.19)$$

where

$$\tilde{F}(\Gamma_0, \Gamma_1, \ldots, \Gamma_{k-1}) = F(\Gamma_1, \Gamma_2, \ldots, \Gamma_k) \quad (3.20)$$

In the thermodynamic limit we get, for fixed F,

$$\langle F\rangle^{\mathrm{DAF}} = \langle \tilde{F}\rangle^{\mathrm{RF}} \tag{3.21}$$

thus establishing the equivalence of the dilute antiferromagnet and the random field models in the hierarchical version.

Our result should also display the physics of the equivalence beyond the hierarchical approximation: summing over contour configurations on a smaller scale with fixed contours in the higher scales should transform, in the limit of infinitely many such steps, the dilute antiferromagnet into a random field Ising model.

4. THE RANDOM FIELD HIERARCHICAL MODEL WITHOUT INTERNAL CONTOURS

In this section we revisit the Bricmont–Kupiainen[5] hierarchical random field model with the extra restriction of absence of internal Peierls contours. From the above analysis the same conclusions should apply to the dilute antiferromagnetic model also taken in its hierarchical version with the provisos mentioned in Section 2. The Hamiltonian is given by

$$H_N^{\mathrm{RF}}(\Gamma) = 2J \sum_{n=0}^{N} L^{n(d-1)} \sum_{x\in\Lambda_n} \gamma_n(x) + \sum_{x\in\Lambda} h_x \sigma_x(\Gamma) \tag{4.1}$$

where $\mathbf{h} = \{h_x, x\in\mathbf{Z}^d\}$ are independent identically distributed random variables with $\overline{h_x} = 0$ and $\overline{h_x^2} = h^2$. For simplicity we shall take h_x with a Gaussian distribution. Notice that the definition (3.3) of $\sigma_x(\Gamma)$ implies that for finite N our model corresponds to boundary conditions $+1$.

For a given realization $\mathbf{h}$ let $\Gamma_G(\mathbf{h})$ denote the configuration of contours that minimizes $H_N^{\mathrm{RF}}(\Gamma)$, i.e., $\Gamma_G(\mathbf{h})$ is the ground state of the model. Let us now estimate the probability that $\sigma_0(\Gamma_G(\mathbf{h})) = -1$. Clearly

$$\begin{aligned}\mathbf{P}\{\sigma_0(\Gamma_G(\mathbf{h})) = -1\} &\leqslant \mathbf{P}\{\exists n \geqslant 0 \text{ such that } \gamma_n(0)\in\Gamma_G(\mathbf{h})\}\\ &= \sum_{n=0}^{N} \mathbf{P}\{\gamma_n(0)\in\Gamma_G(\mathbf{h})\}\end{aligned} \tag{4.2}$$

where in the right-hand side we have the probability that the configuration $\Gamma_G(\mathbf{h})$ contains a contour enclosing the origin and the equality sign is a consequence of the exclusion of internal contours. Now,

$$\gamma_n(0)\in\Gamma_G(\mathbf{h}) \Rightarrow \varepsilon_n \equiv 1 + \frac{1}{J} L^{(1-d/2)n} h(n) \leqslant 0 \tag{4.3}$$

where

$$h(n) \equiv L^{-(d/2)n} \sum_{x \in \gamma_n(0)} h_x \tag{4.4}$$

is a Gaussian random variable with zero mean and variance h, so that, with the use of standard bounds for Gaussian distributions (see, for instance, ref. 10), we get

$$\mathbf{P}\{\gamma_n(0) \in \Gamma_G(\mathbf{h})\} \leqslant f(aL^{(d/2-1)n}) \tag{4.5}$$

where $a = J/h$, and

$$f(a) \equiv \frac{1}{a(2\pi)^{1/2}} e^{-a^2/2} \tag{4.6}$$

If $d > 2$, (4.5) implies the bound

$$\mathbf{P}\{\sigma_0(\Gamma_G(\mathbf{h})) = -1\} \leqslant \sum_{n=0}^{\infty} f(aL^{(d/2-1)n}) \equiv g(a) \tag{4.7}$$

which is uniform in N. Since $g(a) \to 0$ as $a \to \infty$, from (4.7) we see that

$$\overline{\sigma_0(\Gamma_G(\mathbf{h}))} > 0 \tag{4.8}$$

is a is sufficiently large.

For $d = 2$ the estimate (4.5), which is based on the implication (4.3), is not sufficient since it does not decay with n. There is, however, another mechanism repressing the appearance of $\gamma_n(0)$ in $\Gamma_G(\mathbf{h})$: it comes from the exclusion of internal contours and it goes as follows. From (4.5) we see that if $a \gg 1$, the probability of occurrence of small contours in the ground state [i.e., $\gamma_0(x) \in \Gamma_G(\mathbf{h})$] is small. However, for large n we are going to show that with probability that converges to one as $n \to \infty$, it is energetically more favorable to insert the collection of contours $\gamma_0^{(n)}(0) \equiv \{\gamma_0(x), x \in \gamma_n(0), x \neq 0$ such that $h_x \geqslant J\}$ than to insert the contour $\gamma_n(0)$. To make these statements more precise, let for $n \geqslant 1$

$$\Gamma = \gamma_n(0) \cup \bar{\Gamma} \qquad \text{and} \qquad \tilde{\Gamma} = \gamma_0^{(n)}(0) \cup \bar{\Gamma}$$

where $\bar{\Gamma}$ denotes an arbitrary configuration of contours such that Γ (and therefore $\tilde{\Gamma}$) satisfy the condition of "no internal contours." We now compute the difference of energy of the two configurations:

$$\begin{aligned} H_N^{\mathrm{RF}}(\Gamma) - H_N^{\mathrm{RF}}(\tilde{\Gamma}) &= 2JL^n - 2h_0 - 2J \sum_{x \in \gamma_n'(0)} \zeta_x - 2 \sum_{x \in \gamma_n'(0)} h_x(1 - \zeta_x) \\ &\equiv 2JL^n \Delta_n \end{aligned} \tag{4.9}$$

where

$$\zeta_x = \begin{cases} 1 & \text{if } h_x \geqslant J \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

and $\gamma_n'(0) = \gamma_n(0) \setminus \{0\}$, so that the summation is taken over all sites contained in $\gamma_n(0)$ except for the origin; moreover,

$$\Delta_n = 1 - \frac{h_0}{JL^n} - \sum_{x \in \gamma_n'(0)} \left[\frac{1}{L^n} \zeta_x + \frac{1}{JL^n} h_x (1 - \zeta_x) \right] \tag{4.11}$$

Since

$$\gamma_n(0) \in \Gamma_G(\mathbf{h}) \Rightarrow \Delta_n \leqslant 0$$

we have

$$\mathbf{P}\{\gamma_n(0) \in \Gamma_G(\mathbf{h})\} \leqslant \mathbf{P}\{\Delta_n \leqslant 0\} \tag{4.12}$$

The heuristic reason why this mechanism works is based on the computation of the mean value of Δ_n:

$$\overline{\Delta_n} = 1 - L^n \left[\overline{\zeta_0 + \frac{h_0}{J}(1 - \zeta_0)} \right] = 1 + L^n \phi(a) \tag{4.13}$$

where

$$\phi(a) = \left[\overline{\left(\frac{h_0}{J} - 1 \right) \zeta_0} \right] = \frac{1}{(2\pi)^{1/2}} \int_{x \geqslant a} \left[\frac{x}{a} - 1 \right] e^{-x^2/2} \, dx \tag{4.14}$$

Notice that $\phi(a) > 0$ for all $a > 0$. This shows that $\overline{\Delta_n}$ is large for large n. Fluctuations are controlled by standard large deviation results: using Tchebycheff's inequality, we have for $t > 0$

$$\mathbf{P}\{\Delta_n \leqslant 0\} \leqslant \frac{\overline{\exp - t(\Delta_n - \overline{\Delta_n})}}{\exp t\overline{\Delta_n}} \tag{4.15}$$

We now compute the right hand side using the independence of the random fields:

$$\overline{\exp - t(\Delta_n - \overline{\Delta_n})} = \exp\left(\frac{t^2}{2a^2 L^{2n}} \right) \left\{ \overline{\exp \frac{t}{L^n} \left[\zeta_0 + \frac{1}{J} h_0 (1 - \zeta_0) \right]} \right\}^{L^{2n} - 1} \tag{4.16}$$

to obtain

$$\overline{\exp - t(\Delta_n - \overline{\Delta_n})} \leqslant \exp\left(\frac{t^2}{2a^2L^{2n}}\right)\left\{\exp\frac{t^2}{2}c^2\right\} \tag{4.17}$$

where we used the fact that, if we take t sufficiently large depending on $\overline{\zeta_0}$ the following inequality holds

$$\overline{\exp t\left[\zeta_0 + \frac{1}{J}h_0(1-\zeta_0)\right]} \leqslant \exp\left(\frac{c^2t^2}{2}\right) \tag{4.18}$$

It follows from (4.17) that

$$\frac{\overline{\exp - t(\Delta_n - \overline{\Delta_n})}}{\exp t\overline{\Delta_n}} \leqslant \exp - t\overline{\Delta_n}\exp\left(\frac{t^2}{2a^2L^{2n}}\right)\exp\frac{t^2}{2}c^2 \leqslant \exp - t\overline{\Delta_n}\exp\frac{t^2}{2}c^2 \tag{4.19}$$

Let us now choose t minimizing the exponent in r.h.s. i.e.

$$t = \frac{\overline{\Delta_n}}{c^2} \tag{4.20}$$

Notice that for large n this enforces t to be large, so that (4.18) holds. This implies, for large n

$$\mathbf{P}\{\Delta_n \leqslant 0\} \leqslant \exp - \frac{1}{2c^2}(\overline{\Delta_n})^2 \tag{4.21}$$

for some constant c. Combining this bound with the Imry–Ma bound (4.5) we get:

$$\begin{aligned}\mathbf{P}\{\sigma_0(\Gamma_G(\mathbf{h})) = -1\} &\leqslant \sum_0^\infty \mathbf{P}\{\gamma_n(0) \in \Gamma_G(\mathbf{h})\} \\ &\leqslant \overline{\zeta_0} + N_0 f(a) + f(a)^{1/2}\sum_{N_0}^\infty \exp - \frac{1}{4c^2}(\overline{\Delta_n})^2\end{aligned} \tag{4.22}$$

which may be made arbitrarily small if we take a sufficiently large. Here N_0 is chosen so that (4.18) holds for $n \geqslant N_0$. This completes the proof that the ground state exhibits spontaneous magnetization if $d \geqslant 2$ and a is sufficiently large.

Let us now consider the system at non-zero temperature. For a given realization of the random parameters $\mathbf{h}$, the Gibbsian probability that $\sigma_0 = -1$, is given by:

$$p_N = \frac{1}{Z_N} \sum_{n=0}^{N} \sum_{\Gamma \prec \gamma_n(0)} \exp\{-\beta H_N^{RF}(\gamma_n(0) \cup \bar{\Gamma})\} \tag{4.23}$$

where the summation over $\bar{\Gamma} \prec \gamma_n(0)$, means that it is to be taken over all contours Γ, which are compatible with $\gamma_n(0)$. The above formula follows from the fact that $\sigma_0(\Gamma') = -1$ implies that $\Gamma' = (\gamma_n(0) \cup \bar{\Gamma})$ with $\bar{\Gamma} \prec \gamma_n(0)$. Therefore

$$p_N \leqslant \sum_{n=0}^{\infty} r_n \tag{4.24}$$

where

$$r_n = \begin{cases} \exp\{-\beta 2 L^{(d-1)n} J \varepsilon_n\} & \text{if } \varepsilon_n \geqslant 0 \\ 1 & \text{otherwise} \end{cases} \tag{4.25}$$

We now compute averages over $\mathbf{h}$ using the identity

$$\overline{r_n} = \int_0^1 \mathbf{P}\{r_n \geqslant x\}\, dx \tag{4.26}$$

so that

$$\overline{r_n} = \int_0^{\infty} \mathbf{P}\{\alpha_n \varepsilon_n \leqslant u\} \exp(-u)\, du \tag{4.27}$$

where $\alpha_n = 2\beta J L^{(d-1)n}$. From (4.27) it follows that

$$\begin{aligned} \overline{r_n} &\leqslant \int_{\alpha_n/2}^{\infty} \exp(-u)\, du + \int_0^{\alpha_n/2} \mathbf{P}\{\alpha_n \varepsilon_n \leqslant u\} \exp(-u)\, du \\ &\leqslant e^{-\alpha_n/2} + \frac{\alpha_n}{2} \mathbf{P}\left\{\varepsilon_n \leqslant \frac{1}{2}\right\} \end{aligned} \tag{4.28}$$

which together with the standard bound for gaussian distribution leads to

$$\overline{r_n} \leqslant e^{-\alpha_n/2} + \frac{\alpha_n}{2} f\left(\frac{a}{2} L^{(d/2-1)n}\right) \leqslant \alpha_n f\left(\frac{a}{2} L^{(d/2-1)n}\right) \tag{4.29}$$

From this and (4.24) we finally get

$$\overline{p_N} \leqslant \sum_{n=0}^{\infty} \overline{r_n} \tag{4.30}$$

which proves spontaneous magnetization for $d>2$, if a and β are sufficiently large.

For $d=2$ we must incorporate the above discussed refinement for the ground-state to a Peierls' type of argument. We first notice that with the same methods leading to (4.21) we obtain for $0<\delta<1$, $n\geqslant N_0$ (so that (4.18) holds):

$$\mathbf{P}\{\Delta_n\leqslant\delta\}\leqslant\exp-\frac{1}{2c^2}(\overline{\Delta_n}-\delta)^2 \tag{4.31}$$

In an analogous way we obtain:

$$p_N\leqslant\sum_{n=0}^{N_0} r_n+\sum_{n>N_0} s_n \tag{4.32}$$

where for $n\geqslant 1$,

$$s_n=\begin{cases}\exp\{-2\beta JL^n\Delta_n\} & \text{if } \Delta_n\geqslant 0\\ 1 & \text{otherwise}\end{cases} \tag{4.33}$$

We now estimate the average of s_n:

$$\overline{s_n}=\int_0^\infty \mathbf{P}\{\alpha_n\Delta_n\leqslant u\}\exp(-u)\,du \tag{4.34}$$

through

$$\begin{aligned}\overline{s_n}&\leqslant\int_{\delta\alpha_n}^\infty \exp(-u)\,du+\int_0^{\delta\alpha_n}\mathbf{P}\{\alpha_n\Delta_n\leqslant u\}\exp(-u)\,du\\ &\leqslant e^{-\delta\alpha_n}+\delta\alpha_n\mathbf{P}\{\Delta_n\leqslant\delta\}\end{aligned} \tag{4.35}$$

where $0<\delta<1$. Using the bound (4.31) we get for $n>N_0$,

$$\overline{s_n}\leqslant e^{-\delta\alpha_n}+\delta\alpha_n\exp-\frac{1}{2c^2}(\overline{\Delta_n}-\delta)^2\equiv K_n \tag{4.36}$$

so that

$$\overline{p_N}<N_0\alpha_{N_0}f(a)+\sum_{n>N_0}K_n \tag{4.37}$$

thus implying spontaneous magnetization for large a and β.

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